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ON CONTINUOUS DEPENDENCE OF GENERALIZED SOLUTIONS OF THE EQUATION OF UNSTEADY FILTRATION ON A FUNCTION DETERMINING THE FLOW MODE

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The first boundary value problem for the equation of one-dimensional unsteady filtration in a finite region is considered. An estimate is derived in an integral norm for the difference between two generalized solutions corresponding to two different functions relating the pressure with density. The estimate is presented in terms of the maximum value of the modulus of the difference between these two functions.

We shall consider the first boundary value problem in the rectangle $Q = \{(t, x): 0 < t < T,\ 0 < x < l\}$ for the equation

$$u_t = [\varphi(u)]_{xx} \tag{1}$$

where the function $\varphi(u)$ is defined for the values $u \geqslant 0$, and we have $\varphi'(u) > 0$ when $u > 0$. The above equation describes the motion of liquids and gases in a porous medium, and the propagation of heat in a medium with temperature-dependent heat conductivity.

We shall investigate the dependence of the generalized solution of the boundary value problem on the function φ. The dependence of the solutions on the coefficients was studied for the linear parabolic equations in e.g. [1,2], which contain further references. The problem was studied in [3] for a class of nonlinear parabolic equations to which (1) belongs, for the case in which $\varphi'(u) \geqslant \alpha > 0$ for all admissible u. In the present paper we shall deal with the case when $\varphi'(0) = \varphi(0) = 0$.

Let us introduce the notation $\Gamma = \overline{Q} \setminus (Q \cup \{t = T\})$, $\Gamma_1 = \overline{Q} \setminus (Q \cup \{t = 0\})$ and let the following boundary conditions be specified on Γ:

$$u(0, x) = h(x), \quad 0 \leqslant x \leqslant l; \quad u(t, 0) = u(t, l) = 0, \ 0 \leqslant t \leqslant T \tag{2}$$

We shall assume that $h(x) \in C([0, l])$, $0 \leqslant h(x) \leqslant M_0$, $h(0) = h(l) = 0$, $\varphi(h(x))$ satisfy the Lipschitz condition on the segment $0 \leqslant x \leqslant l$, $\varphi(u) \in C^1([0, M_0 + 1]) \cap C^6((0, M_0 + 1])$, $\varphi''(u) > 0$ when $u > 0$. Let us set

$$\Phi(u, v) = \int_0^1 \varphi'(\theta u + (1 - \theta) v)\, d\theta$$

We shall assume that the following inequality holds for any $u \geqslant 0$, $v \geqslant 0$:

$$[\varphi(v)]^p \leqslant M_1 \Phi(u, v), \quad p = \text{const} \in (0, 2), \quad M_1 = \text{const} > 0 \tag{3}$$

The restrictions imposed on φ hold e.g. when $\varphi(u) = u^m$, $m > 1$. In particular, it can be shown that in this case we have (3) in which $p = (m - 1) / m$ and $M_1 = 1$.

Definition. We shall call a generalized solution of the problem (1), (2) the nonnegative function $u(t, x) \in C(\overline{Q})$, which has a bounded generalized derivative

$[\varphi(u(t, x))]_x$, satisfies the conditions (2) and the integral identity

$$\iint_Q \{u f_t - [\varphi(u)]_x f_x\}\, dx\, dt + \int_0^l h(x) f(0, x)\, dx = 0 \tag{4}$$

for any function $f(t, x) \in C^1(\overline{Q})$ vanishing on Γ_1.

It was shown in [1] (see Theorem 3 and Note 1) that under the above assumptions a generalized solution $u(t, x)$ of the problem (1),(2) exists and is unique. In addition,

$$0 \leqslant u(t, x) \leqslant M_0 \text{ everywhere in } \overline{Q} \tag{5}$$

Let the function $\psi(u)$ have the same properties as $\varphi(u)$ except, perhaps, (3). Let in addition

$$\psi(u) \leqslant M_2 \varphi(u), \quad M_2 = \text{const} > 0 \tag{6}$$

We denote by $v(t, x)$ the generalized solution of the first boundary value problem in the rectangle Q for the equation

$$u_t = [\psi(u)]_{xx}$$

and conditions (2). Let us set $a(t, x) = \Phi(u(t, x), v(t, x))$.

Theorem. The following estimate holds:

$$\iint_Q [a(t, x)]^{1/4} [u(t, x) - v(t, x)]^2\, dx\, dt \leqslant K \max_{0 \leqslant s \leqslant M_0} |\psi(s) - \varphi(s)|^{1-p/2} \tag{7}$$

where the constant $K > 0$ depends only on the quantities

$$M_0, M_1, M_2, p, T, l \tag{8}$$

Proof. By virtue of (2) and (4) we have

$$\iint_Q [u f_t + \varphi(u) f_{xx}]\, dx\, dt + \int_0^l h(x) f(0, x)\, dx = 0 \tag{9}$$

Writing an analogous identity for the function $v(t, x)$ and subtracting it from (9), we obtain

$$\iint_Q \{(u - v)(f_t + a f_{xx}) + [\varphi(v) - \psi(v)] f_{xx}\}\, dx\, dt = 0 \tag{10}$$

Let us construct a monotonously decreasing sequence of the positive functions $a_n(t, x) \in C^\infty(\overline{Q})$ $(n = 1, 2, \ldots)$, converging uniformly in $\overline{Q}$ to $a(t, x)$ as $n \to \infty$. Let us also construct a sequence of functions $z_n(t, x) \in C_0^\infty(\overline{Q})$ $(n = 1, 2, \ldots)$ converging uniformly in $\overline{Q}$ to $u(t, x) - v(t, x)$ as $n \to \infty$. As we know, (see e.g. [5]) for every $n = 1, 2, \ldots$ there exists a unique solution $f = f_n(t, x) \in C^{1,2}_{t,x}(\overline{Q})$ of the first boundary value problem for the equation

$$f_t + a_n f_{xx} = a_n^{1/2} z_n \tag{11}$$

in Q, with the condition that

$$f = 0 \quad \text{on } \Gamma_1 \tag{12}$$

Let us multiply both parts of (11) by f_{nxx}, integrate the resulting equation over Q, integrate by parts with (1.2) taken into account, and apply the Cauchy inequality.

This yields

$$\iint_Q a_n f_{nxx}^2\, dx\, dt = \iint_Q a_n^{1/2} f_{nxx} z_n\, dx\, dt - \frac{1}{2}\int_0^l f_n^2(0, x)\, dx \leqslant$$

$$\frac{1}{2}\iint_Q a_n f_{nxx}^2\, dx\, dt + \frac{1}{2}\iint_Q z_n^2\, dx\, dt$$

from which we obtain

$$\iint_Q a_n f_{nxx}^2\, dx\, dt \leqslant \iint_Q z_n^2\, dx\, dt \leqslant K_1 \tag{13}$$

Here and henceforth K_i will denote positive constants depending on the qualities (8) only.

Substituting $f = f_n$ into (10) yields

$$\iint_Q (u - v)\, a_n^{1/2} z_n\, dx\, dt = \iint_Q (u - v)(a_n - a) f_{nxx}\, dx\, dt + \tag{14}$$

$$\iint_Q [\psi(v) - \varphi(v)] f_{nxx}\, dx\, dt \equiv I_{1n} + I_{2n}$$

We estimate I_{1n} using the Cauchy — Buniakowski inequality and the relations (13)and (5). We have

$$I_{1n}^2 \leqslant \iint_Q a_n f_{nxx}^2\, dx\, dt \iint_Q (u - v)^2 (1 - a_n^{-1} a)(a_n - a)\, dx\, dt \leqslant \tag{15}$$

$$K_2 \max_{\overline{Q}} (a_n - a) \to 0 \quad (n \to \infty)$$

The integral I_{2n} can be estimated with the help of the inequalities (13), (3), (5) and (6). We denote by E the set of points $(t, x) \in Q$ in which $v(t, x) > 0$. We have

$$I_{2n}^2 \leqslant K_1 \iint_E a^{-1} [\psi(v) - \varphi(v)]^2\, dx\, dt \leqslant \tag{16}$$

$$K_3 \iint_E \{[\psi(v) + \varphi(v)] / \varphi(v)\}^p\, |\psi(v) - \varphi(v)|^{2-p}\, dx\, dt \leqslant$$

$$K_4 \max_{0 \leqslant s \leqslant M_0} |\psi(s) - \varphi(s)|^{2-p}$$

Passing now to the limit in (14) with $n \to \infty$ and taking into account the relations (15) and (16), we arrive at the required estimate (7).

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